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1998 J. Phys. A: Math. Gen. 31 6647

(http://iopscience.iop.org/0305-4470/31/31/011)

J. Phys. A: Math. Gen. **31** (1998) 6647–6652. Printed in the UK PII: S0305-4470(98)93487-4

A note on the generalized Lie algebra $sl(2)_q$

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Received 20 April 1998

Abstract. In a recent paper, Dobrev and Sudbery classified the highest weight and lowest weight finite-dimensional irreducible representations of the quantum Lie algebra $sl(2)_q$ introduced by Lyubashenko and Sudbery. The aim of this note is to add to this classification all the finite-dimensional irreducible representations which have no highest weight and/or no lowest weight, in the case when q is a root of unity. For this purpose, we give a description of the enlarged centre.

1. Introduction

In the notion of 'quantum groups' introduced by Drinfeld and Jimbo [1, 2], one actually refers to the quantization of the enveloping algebra $\mathcal{U}(\mathcal{G})$, considered as a Hopf algebra. The question arises about the existence of a deformation of the Lie algebra itself, and several authors have more recently made progresses towards a definition of quantized Lie algebras [3–5].

In [6], Dobrev and Sudbery gave a classification of finite-dimensional irreducible representations of the quantum Lie algebra $sl(2)_q$ as defined in [5] by Lyubashenko and Sudbery. This classification actually concerns the highest weight and lowest weight representations. It happens, however, that there exists other classes of finite-dimensional representations of quantum groups at roots of unity that are useful for physics, namely the periodic (cyclic) representations [7, 8], which appear for instance in generalizations of the chiral Potts model [9, 10].

The quantum Lie algebra is defined as a finite-dimensional subspace of the quantized enveloping algebra that is invariant under the quantum adjoint action. According to [5], the representation theory of $sl(2)_q$ reduces to that of the algebras $\mathcal{B}$ and $\mathcal{F}$ defined below.

The algebra $\mathcal{B}$ is generated by X_0, $X_\pm$, C, related by

$$q^2 X_0 X_+ - X_+ X_0 = qCX_+ \tag{1}$$

$$q^{-2} X_0 X_- - X_+ X_0 = -q^{-1}CX_- \tag{2}$$

$$X_+X_- - X_-X_+ = (q+q^{-1})(C-\lambda X_0)X_0 \tag{3}$$

$$CX_\pm - X_\pm C = CX_0 - X_0 C = 0 \tag{4}$$

where $\lambda = q - q^{-1}$. We will later use q-numbers $[p]$ defined as usual by $[p] \equiv \frac{q^p - q^{-p}}{q-q^{-1}}$.

A quadratic central element of $\mathcal{B}$ is given by

$$C_2' = X_-X_+ + q^{-1}CX_0 + q^{-2}X_0^2 \tag{5}$$

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0305-4470/98/316647+06$19.50

(normalized by a factor of $(q+q^{-1})^{-1}$ with respect to [6]).

The algebras $\mathcal{F}$ and $\mathcal{A}$ are defined from $\mathcal{B}$ by adding respectively the relations $C^2 - \lambda^2 C_2' = 1$ and $C = 1$ on central elements [5].

When interpreted in the $\mathcal{U}_q(sl(2))$ context, C corresponds to the usual quadratic Casimir element, whereas the quadratic central element C_2' of $\mathcal{B}$ corresponds to a quartic central element [5].

The classification of finite-dimensional irreducible representations of $\mathcal{U}_q(sl(2))$ at roots of unity (including periodic ones) was given in [8]. The classification we present here is very close to the latter. The representations of $\mathcal{B}$ have one more parameter. Unusual representations of dimension 1 are present.

The methods we use here are similar to those used in [11] in the case of $\mathcal{U}_q(\mathrm{osp}(1|2))$, and more details may be found there.

2. Centre at $q^{2l} = 1$

A linear basis of $\mathcal{B}$ is given by

$$X_-^{a_-} X_+^{a_+} X_0^{a_0} C^{b_1} {C_2'}^{b_2} \qquad \text{with } a_\pm, a_0, b_1, b_2 \in \mathbb{N}, a_+ a_- = 0. \tag{6}$$

This can be proved starting from a basis of the form given in [5], lemma 3.2. Then all the common powers of X_- and X_+ in a monomial can indeed be re-expressed in terms involving X_0, C and C_2' only, using

$$X_- X_+ = C_2' - q^{-1} C X_0 - q^{-2} X_0^2 \tag{7}$$

$$= \lambda^{-2}\{-(C^2 - \lambda^2 C_2') + (1+q^{-2}) C (C - \lambda X_0) - q^{-2} (C - \lambda X_0)^2\} \tag{8}$$

$$X_+ X_- = C_2' + q C X_0 - q^2 X_0^2 \tag{9}$$

$$= \lambda^{-2}\{-(C^2 - \lambda^2 C_2') + (1+q^{2}) C (C - \lambda X_0) - q^{2} (C - \lambda X_0)^2\}. \tag{10}$$

The centre of $\mathcal{B}$ for generic q is generated by C and C_2'. (A linear combination of terms given by (6) needs, in order to commute with X_0, to involve only terms with $a_+ = a_- = 0$. In order to commute with $X_\pm$, it should not involve terms with $a_0 \neq 0$.)

2.1. Letting q be a root of unity

More precisely let l be the smallest (non-zero) integer such that $q^{2l} = 1$.

The centre of $\mathcal{B}$ is now $\mathbb{C}[C, C_2', X_+^l, (C - \lambda X_0)^l] + \mathbb{C}[C, C_2', X_-^l, (C - \lambda X_0)^l]$. The sum is not a direct sum and the intersection is $\mathbb{C}[C, C_2', (C - \lambda X_0)^l]$.

The generators C, C_2', $X_\pm^l$ and $(C - \lambda X_0)^l$ of the centre of $\mathcal{B}$ when $q^{2l} = 1$ are subject to the relation

$$X_-^l X_+^l = q^{l(l-1)} \lambda^{-2l} \{-(\mathcal{D}^2)^l + q^{-l} \mathcal{D}^l \mathcal{Q}_l((q+q^{-1}) C \mathcal{D}^{-1}) (C - \lambda X_0)^l - (C - \lambda X_0)^{2l}\} \tag{11}$$

where $\mathcal{Q}_l$ is the polynomial of degree l, related to the Chebichev polynomial of the first kind, such that

$$\mathcal{Q}_l(x + x^{-1}) = x^l + x^{-l} \tag{12}$$

and where $\mathcal{D}$ is defined by

$$\mathcal{D}^2 = C^2 - \lambda^2 C_2'. \tag{13}$$

Note that the right-hand side of (11) is a well-defined polynomial of degree l in $\mathcal{D}^2$, and hence in C_2'.

To prove formula (11), we proceed as in [12]: we first prove by a simple recursion

$$X_-^p X_+^p = \lambda^{-2p} \prod_{r=0}^{p-1} q^{-2r-1}\{-q\mathcal{D}^2 q^{2r} + (q+q^{-1})C(C-\lambda X_0) - q^{-1}(C-\lambda X_0)^2 q^{-2r}\} \quad (14)$$

and then let $p = l$, so that the operand runs over all the powers of q^2.

3. Finite-dimensional irreducible representations of $\mathcal{B}$

We now give the classification of finite-dimensional irreducible representations when $q^{2l} = 1$, insisting on those with no highest weight (and/or lowest weight) vector, which were not considered in [6]. We use module notations.

On any finite-dimensional simple module, the central elements C, C_2', $X_\pm^l$ and $(C-\lambda X_0)^l$ act as scalars (diagonal matrices with a single eigenvalue), which we denote respectively by c, c_2', $x_\pm^l$ and z, and which satisfy the relation (obtained from (11))

$$x_-^l x_+^l = q^{l(l-1)}\lambda^{-2l}\{-(d^2)^l + q^{-l}d^l \mathcal{Q}_l((q+q^{-1})cd^{-1})z - z^2\} \quad (15)$$

where $d^2 \equiv c^2 - \lambda^2 c_2'$. Note that (15) is a polynomial of degree l in d^2, and hence in c_2'.

Let M be a finite-dimensional simple module. There exists in M a vector v_0 such that, in addition to $Cv_0 = cv_0$, $C_2'v_0 = c_2'v_0$, $X_\pm^l v_0 = x_\pm^l v_0$ and $(C-\lambda X_0)^l v_0 = zv_0$, we also have

- $X_0 v_0 = x_0 v_0$ with $z = (c - \lambda x_0)^l$,
- $M = \mathrm{span}\{X_+^p v_0, X_-^p v_0\}_{p=0,\ldots,l-1}$ (these vectors being linearly dependent).

The existence of v_0 satisfying the first property is guaranteed by the finite dimension. The second property is proved by writing $M = \mathcal{B}.v_0$, using the basis (6), and observing that $X_\pm^p v_0$ are eigenvectors of C, C_2' and X_0.

3.1. First case: $z \neq 0$ and $x_- \neq 0$ (X_- acts injectively)

We define

$$v_p = x_-^{-p} X_-^p v_0 \qquad (v_l \equiv v_0). \quad (16)$$

Then

$$X_0 v_p = (q^{2p}x_0 - q^p[p]c)v_p \quad (17)$$

$$X_- v_p = x_- v_{p+1} \quad (18)$$

$$X_+ v_p = x_-^{-1}\lambda^{-2}\{-d^2 + (1+q^{-2})c(c-\lambda x_0)q^{2p} - q^{-2}(c-\lambda x_0)^2 q^{4p}\}v_{p-1}. \quad (19)$$

The action of X_+ on v_p is computed using $X_+ v_p = x_-^{-1} X_+ X_- v_{p-1}$ and equation (10). The module spanned by v_p, $p = 0, \ldots, l-1$ is simple since the eigenvectors v_p of X_0 correspond to l different eigenvalues (this would not be the case with $z = 0$).

This class of periodic (or semiperiodic when $x_+ = 0$) l-dimensional representations is then characterized by the five complex parameters c, c_2', $x_\pm^l$ and z, related by the polynomial relation (15).

3.2. Second case: $z \neq 0$, $x_- = 0$ and $x_+ \neq 0$ (X_+ acts injectively, but not X_-)

This case is symmetric to a subcase of the previous one, for which $x_+ = 0$ was not excluded.

Let $w_0 = v_0$ (such that $X_0 w_0 = x_0 w_0$) requiring further that $X_- w_0 = 0$. Such a vector exists because: (i) X_- is nilpotent; (ii) the eigenspace of X_- related to the eigenvalue 0 is stable under the action of X_0. We have $c_2' = q^2 x_0^2 - qcx_0$ and $z = (c - \lambda x_0)^l$. We define

$$w_p = x_+^{-p} X_+^p w_0 \qquad (w_l \equiv w_0). \tag{20}$$

Then

$$X_0 w_p = (q^{-2p} x_0 + q^{-p}[p]c) w_p \tag{21}$$

$$X_+ w_p = x_+ w_{p+1} \tag{22}$$

$$X_- w_p = x_+^{-1} \lambda^{-2} \{-d^2 + (1+q^2)c(c - \lambda x_0) q^{-2p} - q^2 (c - \lambda x_0)^2 q^{-4p}\} w_{p-1}. \tag{23}$$

The action of X_- on w_p is computed using $X_- w_p = x_+^{-1} X_- X_+ w_{p-1}$ and equation (8). The module spanned by w_p, $p = 0, \ldots, l-1$ is again simple.

This class of semiperiodic l-dimensional representations is then characterised by the three complex parameters c, x_0, x_+^l. The parameters c_2' and z are related to those by $c_2' = q^2 x_0^2 - qcx_0$ and $z = (c - \lambda x_0)^l$.

3.3. Third case: $z \neq 0$ and $x_- = x_+ = 0$ (highest weight and lowest weight representation)

This case has been studied in details in [6]. We just give a summary of the classification given there.

- There are one-parameter irreducible representations of dimension $n < l$, described by

$$C v_p = c v_p \tag{24}$$

$$X_0 v_p = \lambda^{-1} \left(c - q^{2p} \nu\right) v_p \tag{25}$$

$$X_- v_p = v_{p+1} \qquad X_- v_{n-1} = 0 \tag{26}$$

$$X_+ v_p = \lambda^{-1} [p] q^{p-2} \nu \{(q^2+1)c - (q^{2p}+1)\nu\} v_{p-1} \tag{27}$$

with the constraint $(q^2+1)c = (q^{2n}+1)\nu$† and $\nu \neq 0$. Note that when $l = 2$, $n = 1$, this is a two-parameter representation.

- There are l-dimensional irreducible representations, also described by (27) (with $n = l$), and characterized by two parameters c and $\nu \neq 0$, with the constraint that $(q^2+1)c - (q^{2p}+1)\nu \neq 0$ for $p = 1, \ldots, l-1$. These representations do not exist when $l = 2$.

3.4. Fourth case: $z = 0$

Supposing first $x_- \neq 0$, we define v_p, $p = 0, \ldots, l-1$ as in the first case. The action of X_0, $X_\pm$ are as in (17)–(19). Now, this defines a reducible representation since all the eigenvalues of X_0 are equal. Irreducible one-dimensional subrepresentations are defined by any vector $v = \sum_{p=0}^{l-1} q^{2kp} v_p$, and

$$\begin{array}{l} Cv = cv \\ C_2' v = c_2' v \\ X_0 v = x_0 v \\ X_\pm v = x_\pm' v \end{array} \qquad \text{with} \quad \begin{array}{l} c - \lambda x_0 = 0 \\ x_\pm' = q^{\pm 2k} x_\pm \\ x_+ x_- = c_2' - \lambda^{-2} c^2 = -\lambda^{-2} d^2. \end{array} \tag{28}$$

Considering then the case $x_- = 0$, $x_+ \neq 0$, and following the construction defined by (20)–(23) again leads to (28). The case $z = x_- = x_+ = 0$, already in the classification of [6], is also described by (28).

† Note that with this parametrization, it is not necessary to distinguish the case $q^{2n} + 1 = 0$, i.e. $n = l/2$, when l is even, for which $c = 0$. This is, however, not true for representations of $\mathcal{A}$, for which $c = 1$.

This class of one-dimensional representations described by (28) is characterized by three continuous parameters x_0, $x'_\pm$.

Remark. Even in the case when q is generic, there exists (semi)periodic representations of dimension 1, given by

$$X_0 v = x_0 v \qquad C v = c v \qquad X_\pm v = x_\pm v \qquad \text{with } c - \lambda x_0 = 0. \tag{29}$$

3.5. Representations of $\mathcal{A}$

The irreducible finite-dimensional representations of $\mathcal{A}$ are given by fixing $c = 1$ in the previous classification. This is generally possible, except for the representations of dimension $l/2$ (when $l/2 \in \mathbb{N}$) for which the constraint was $c = 0$.

4. Finite-dimensional irreducible representations of $\mathcal{F}$

The algebra $\mathcal{F}$ is defined from $\mathcal{B}$ as its quotient by the relation $C^2 = 1 + \lambda^2 C'_2$, i.e. $\mathcal{D}^2 = 1$ (13). One obtains the irreducible finite-dimensional representations of $\mathcal{F}$ from those of $\mathcal{B}$ by imposing the supplementary condition $d^2 = c^2 - \lambda^2 c'_2 = 1$ on the parameters. Generically, the parameters are then c, $x^l_\pm$ and z, eigenvalues of C, $X^l_\pm$ and $(C - \lambda X_0)^l$, related by

$$x^l_- x^l_+ = q^{l(l-1)} \lambda^{-2l} \{-1 + q^{-l} d^l \mathcal{Q}_l((q + q^{-1})c) z - z^2\}. \tag{30}$$

We still consider only the case when q is a root of unity. The classification is then the following.

4.1. First case: $z \neq 0$ and $x_- \neq 0$

The representations with injective action of X_-, of dimension l, are described by (16)–(19) with $d^2 = 1$. They depend on the parameters c, $x^l_\pm$ and z, related by (30).

4.2. Second case: $z \neq 0$, $x_- = 0$ and $x_+ \neq 0$

The representations with nilpotent action of X_- and injective action of X_+, of dimension l, are described by (20)–(23) with $d^2 = 1$. This class of semiperiodic l-dimensional representations depends on the parameters ν, x^l_+, from which c, x_0 and z are given by $c = (q\nu + q^{-1}\nu^{-1})/[2]$, $x_0 = \lambda^{-1}(c - \nu)$ and $z = \nu^l$.

4.3. Third case: $z \neq 0$ and $x_- = x_+ = 0$

This case corresponds to the classification in [6].

- The representations of dimension $n < l$ are described by (27) with $[2]c = q^{-1}\nu + q\nu^{-1}$ and $\nu^2 = q^{-2n+2}$. Hence, they are labelled by the dimension n and a sign ϵ such that $\nu = \epsilon q^{-n+1}$.
- The representations of dimension l are described by (27) with again $[2]c = q^{-1}\nu + q\nu^{-1}$, and now $\nu^2 \neq q^{-2p+2}$ for $p = 1, \ldots, l-1$. They are labelled by one parameter ν.

4.4. Fourth case: $z = 0$

The unusual representations of dimension 1 described by (28) still exist for $\mathcal{F}$, with $d^2 = 1$. These representations are necessarily periodic since $x_+x_- = -\lambda^{-2}$, which explains why they are not in the classification of [6]. They depend on two parameters x_0 and x_+.

Acknowledgment

This work was partially supported by European Community contract TMR FMRX-CT96.0012.

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